STABLE STRUCTURES ON MANIFOLDS

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Recent developments in topology seem to support the observation that a number of essentially local problems in certain manifolds can be solved by ignoring the local nature of the problem and instead using to advantage the global form of the manifold.

It is the object of this report to describe a certain structure on manifolds, called a *stable structure*, which capitalizes on this observation. Orientable differentiable and piecewise linear manifolds, as well as all simply connected topological manifolds, support stable structures, and one can solve in such manifolds a number of problems, both local and global in nature, which remain unsolved in general.

The three parts of this report correspond to the papers [**1**; **2**; **3**], in which the various details may be found. The first part deals with the homeomorphisms of the n-sphere and describes some results necessary for the development of the machinery of stable structures. The second part introduces stable structures and shows that certain theorems proved for the n-sphere will hold for an arbitrary connected topological manifold if and only if that manifold supports a stable structure. The third part applies the machinery of stable structures to some problems in the field of topological manifolds.

I. Homeomorphisms of S^n

1. Definitions. The set of points $\{(x_1, \cdots, x_n): \sum x_i^2 \leqq 1\}$ in Euclidean n-space R^n will be denoted by D^n and its boundary by S^{n-1}. D^n and any space homeomorphic to D^n will be called a *closed n-cell.* S^{n-1} and any space homeomorphic to S^{n-1} will be called an *$n-1$-sphere.*

$H(S^n)$ will denote the group of homeomorphisms of S^n onto itself.

A k-manifold M^k in an n-manifold M^n will be said to be *locally flat* if each point of M^k has a neighborhood U in M^n such that the pair $(U, U\cap M^k)$ is topologically equivalent to the pair (R^n, R^k). Then $\operatorname{Hom}(S^{n-1}, S^n)$ will denote the set of all locally flat embeddings of S^{n-1} into S^n.

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2. Annular equivalence of embeddings of S^{n-1} in S^n. Hom(S^{n-1}, S^n) is studied first and then the information obtained is used to study $H(S^n)$.

Let f_0 and f_1 be elements of Hom(S^{n-1}, S^n). If there is an *embedding* $F: S^{n-1}\times[0, 1]\rightarrow S^n$ such that, for all $x\in S^{n-1}$, $F(x, 0)=f_0(x)$ and $F(x, 1)=f_1(x)$, then F will be called a *strict annular equivalence* between f_0 and f_1, and we write

$$f_0\underset{A}{\sim}f_1.$$

Then of course f_0 and f_1 must have disjoint images and "similar" orientations.

Strict annular equivalence is not an equivalence relation, but generates one as follows. Two elements f and f' of Hom(S^{n-1}, S^n) will be said to be *annularly equivalent*, written

$$f\underset{a}{\sim}f',$$

if there is a finite sequence of elements $f=f_0, f_1, \cdots, f_k=f'$ of Hom(S^{n-1}, S^n) such that

$$f_i\underset{A}{\sim}f_{i+1}$$

for $i=0, 1, \cdots, k-1$. Annular equivalence is an equivalence relation.

The main result about annual equivalence in S^n is

THEOREM 2.1. *Let f and f' be annularly equivalent elements of* Hom(S^{n-1}, S^n) *with disjoint images and similar orientations. Then*

$$f\underset{A}{\sim}f'.$$

3. Stable homeomorphisms. Let h be a homeomorphism of S^n onto itself. If there is a nonempty open set $U\subset S^n$ such that $h/U=1$, we will say that h is *somewhere the identity*. Then $SH(S^n)$, the group of *stable homeomorphisms* of S^n, will consist of products of homeomorphisms, each of which is somewhere the identity.

It is shown in [4] that $SH(S^n)$ is the intersection of all nontrivial normal subgroups of $H(S^n)$ and is, furthermore, simple. In particular, $SH(S^n)$ must be arcwise connected in the compact-open topology. Hence every stable homeomorphism is isotopic to the identity through stable homeomorphisms.

It follows easily from [5] that an orientation preserving differentiable homeomorphism of S^n onto itself is stable. Similarly, it follows from [6; 7] that an orientation preserving piecewise linear homeomorphism of S^n onto itself is stable.

Now let f and f' be elements of $\mathrm{Hom}(S^{n-1}, S^n)$. If there is a stable homeomorphism h of S^n onto itself such that $hf=f'$, then we say that f and f' are *stably equivalent*, and write

$$f \underset{s}{\sim} f'.$$

The application of $\mathrm{Hom}(S^{n-1}, S^n)$ to the study of $H(S^n)$ depends on the following result.

THEOREM 3.1. *Two elements of* $\mathrm{Hom}(S^{n-1}, S^n)$ *are stably equivalent if and only if they are annularly equivalent.*

Using this, we obtain

THEOREM 3.2. *Let h be a stable homeomorphism of S^n and E_1, E_2 closed n-cells in S^n with locally flat boundaries. If $E_1 \cup hE_1$ is disjoint from E_2, then there is a stable homeomorphism h' of S^n which agrees with h on E_1 and whose restriction to E_2 is the identity.*

COROLLARY. *Any stable homeomorphism of S^n can be written as the product of two homeomorphisms, each of which is somewhere the identity.*

II. STABLE MANIFOLDS

4. Homeomorphisms of topological manifolds. Let M^n be a connected topological manifold of dimension n. An embedding $f: D^n \to M^n$ will be said to be *locally flat* if f/S^{n-1} is locally flat. Then $\mathrm{Hom}(D^n, M^n)$ will denote the set of all locally flat embeddings of D^n into M^n.

$H(M^n)$ will denote the group of homeomorphisms of M^n onto itself and $SH(M^n)$, the group of *stable homeomorphisms* of M^n, will consist of products of homeomorphisms, each of which is somewhere the identity. If h is a homeomorphism of M^n onto itself and E a closed n-cell in M^n with locally flat boundary, such that $h/M^n - E$ is the identity, then we will say that h is *almost everywhere the identity*. $SH_0(M^n)$ will consist of products of homeomorphisms, each of which is almost everywhere the identity. It is shown in [4] that $SH_0(M^n)$ is the intersection of all nontrivial normal subgroups of $H(M^n)$ and is simple. Note that $SH(S^n) = SH_0(S^n)$.

As in Part I we can define *strict annular equivalence*, *annular equivalence* and *stable equivalence* between elements of $\mathrm{Hom}(D^n, M^n)$. It will again hold true that two elements of $\mathrm{Hom}(D^n, M^n)$ are stably equivalent if and only if they are annularly equivalent, so that $\mathrm{Hom}(D^n, M^n)$ may be used to study $H(M^n)$.

Theorem 3.2 translates word for word to

THEOREM 4.1. *Let h be a stable homeomorphism of M^n and E_1, E_2 closed n-cells in M^n with locally flat boundaries. If $E_1 \cup hE_1$ is disjoint* *from E_2, then there is a stable homeomorphism h' of M^n which agrees with h on E_1 and whose restriction to E_2 is the identity.*

COROLLARY. *Any stable homeomorphism of M^n can be written as the product of two homeomorphisms, each of which is somewhere the identity.*

While Theorem 4.1 seems to be a complete generalization of Theorem 3.2, note that in the case of S^n the theorem can be reworded so that E_2, instead of being disjoint from $E_1 \cup hE_1$, contains $E_1 \cup hE_1$ in its interior. In the case of an arbitrary connected manifold M^n, the reworded theorem is no longer equivalent to the original one.

Note that in the following theorem we are not free to choose E_2.

THEOREM 4.2. *Let h be a stable homeomorphism of M^n and E_1 a closed n-cell in M^n with locally flat boundary, such that E_1 and hE_1 are disjoint. Then there exists a closed n-cell E_2 in M^n with locally flat boundary, which contains $E_1 \cup hE_1$ in its interior, and a stable homeomorphism h' of M^n which agrees with h on E_1 and whose restriction to $M^n - E_2$ is the identity.*

5. **Stable structures.** A homeomorphism f from the open set $U \subset R^n$ onto the open set $V \subset R^n$ will be said to be *stable at* $x \in U$ if there is a neighborhood U_x of x in U and a stable homeomorphism h of R^n onto itself such that $f/U_x = h/U_x$.

The set of points of U at which f is stable is, by definition, open. But a homeomorphism of R^n which agrees with a stable homeomorphism on a nonempty open set must itself be stable. It then follows easily that the set of points of U at which f is not stable is also open. Hence if f is stable at $x \in U$, it must be stable at every point of the component of U containing x.

If f is stable at every point of U, we will say that f is *stable on* U, and call f a *stable coordinate transformation*. The collection of all stable coordinate transformations forms a pseudogroup, which we denote by $SP(R^n)$.

The *sheaf* $S(M^n)$ *of germs of stable structures* on the connected manifold M^n is then constructed in the standard way from the pseudogroup of stable coordinate transformations. Because stability of coordinate transformations occurs on components, it turns out that $S(M^n)$ is a principal bundle over M^n with group and fibre the discrete group $H(R^n)/SH(R^n)$, which, incidentally, is isomorphic to $H(S^n)/SH(S^n)$. Such an object differs from a regular covering space over M^n only in that $S(M^n)$ is not necessarily connected. However, as a principal bundle, the various components of $S(M^n)$ must be equivalent over M^n, and in this generalized sense we state

THEOREM 5.1. *The sheaf $S(M^n)$ of germs of stable structures on M^n is a regular covering space over M^n.*

The well-defined normal subgroup of $\pi_1(M^n)$ corresponding to this regular covering will be called the *stability subgroup* of $\pi_1(M^n)$, and denoted by $S\pi_1(M^n)$.

A global cross section $f: M^n \to S(M^n)$ will be called a *stable structure* on M^n. Then we immediately have

THEOREM 5.2. *Every simply connected manifold admits a stable structure.*

It is clear that M^n admits a stable structure if and only if M^n can be covered by a family of local coordinate systems with stable coordinate transformations on the intersections. Then since orientation preserving differentiable and piecewise linear coordinate transformations are stable, we have

THEOREM 5.3. *Every orientable differentiable or piecewise linear manifold admits a stable structure.*

It is also true that

THEOREM 5.4. *Every orientable triangulable manifold admits a stable structure.*

6. Stable homeomorphisms of stable manifolds. Let M^n be a stable manifold, $f: M^n \to S(M^n)$ a stable structure on M^n, and h a homeomorphism of M^n onto itself. It is tempting to call h *stable* if it "preserves" the stable structure f, but this terminology has already been used in a different sense in §4. The following theorem, however, removes any possibility of confusion.

THEOREM 6.1. *A homeomorphism h of the stable manifold M^n is stable in the new sense if and only if it is stable in the old sense. In particular, the stabilitiy of h is independent of the particular stable structure on M^n.*

For stable homeomorphisms of stable manifolds, the theorems of Part I admit full generalizations.

THEOREM 6.2. *Let M^n be a connected topological manifold. Then the following are equivalent:*

(i) *M^n admits a stable structure.*

(ii) *If h is a stable homeomorphism of M^n and E_1, E_2 closed n-cells in M^n with locally flat boundaries, such that $E_1 \cup hE_1 \subset \operatorname{Int} E_2$, then there is a stable homeomorphism h' of M^n which agrees with h on E_1 and whose restriction to $M^n - E_2$ is the identity.*

(iii) *If f and f' are annularly equivalent elements of* $\operatorname{Hom}(D^n, M^n)$ *such that* $f(D^n) \subset \operatorname{Int} f'(D^n)$, *then*

$$f \underset{A}{\sim} f'.$$

(iv) *If U is a connected open subset of M^n and f and f' are elements of* $\operatorname{Hom}(D^n, U)$ *which are annularly equivalent in M^n, then they are annularly equivalent in U.*

7. **Covering spaces.** Since $S(M^n)$ is a covering space over M^n, it is to be expected that a number of relations will appear between the stable structures on M^n and those on its various covering spaces.

THEOREM 7.1. *Any covering space of a stable manifold is stable.*

THEOREM 7.2. *There is a minimal stable covering space of any given manifold. It corresponds to the stability subgroup and is equivalent to a component of the sheaf of germs of stable structures on the manifold. Hence for any manifold, the sheaf of germs of stable structures is stable.*

THEOREM 7.3. *Every covering transformation of a covering space of a stable manifold is stable.*

The above theorem is used to show that if the annulus conjecture[2] is false in dimension n, there is a closed orientable n-manifold which admits no stable structure. By Theorem 5.3, such a manifold admits neither a differentiable nor a piecewise linear structure. By Theorem 5.4, such a manifold cannot be triangulated.

THEOREM 7.4. *Let $\tilde{M}^n$ be a regular covering space of M^n. Then M^n is stable if and only if $\tilde{M}^n$ is stable and all the covering transformations are stable.*

Combining the above theorem with Theorem 5.2, we get the following characterization of stable manifolds.

THEOREM 7.5. *A connected topological manifold is stable if and only if each covering transformation of its universal covering space can be written as a product of homeomorphisms, each of which is somewhere the identity.*

8. **The homogeneity problem.** The connected n-manifold M^n is said to be *homogeneous* if for any two locally flat embeddings f, f' of D^n into M^n there is a homeomorphism h of M^n onto itself such that $hf = f'$. If M^n is orientable and h exists provided f and f' induce the

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[2] The *annulus conjecture* claims that the closed region between any two disjoint locally flat $n-1$-spheres in S^n is homeomorphic to $S^{n-1} \times [0, 1]$.

same orientation on M^n from a given orientation on D^n, we say that M^n is *homogeneous up to orientation.*

Orientable manifolds which admit no orientation reversing homeomorphism cannot be homogeneous, but it is a classical conjecture that all connected manifolds are homogeneous up to orientation. This conjecture has been proved in the differentiable case by Palais [5] and in the piecewise linear case by Newman [6] and Gugenheim [7].

If M^n admits a stable structure and there is a stable homeomorphism between any two stable manifolds with underlying space M^n, we will say that the stable structure on M^n is *unique.* If the stable homeomorphism exists provided the stable structures induce the same orientation on M^n, then we say that the stable structure on M^n is *unique up to orientation.*

THEOREM 8.1. *Let M^n be a connected stable manifold. Then the stable structure on M^n is unique (unique up to orientation) if and only if M^n is homogeneous (homogeneous up to orientation).*

III. APPLICATIONS

9. **Tubular neighborhoods.** The underlying space of an R^{n-1} bundle over S^1 will be called a *tube.* A tube will be said to be *trivial* if it is homeomorphic to $R^{n-1}\times S^1$. A subset C of the trivial tube T will be called a *core* of T if there is a homeomorphism $F: R^{n-1}\times S^1\rightarrow T$ such that $C=F(0\times S^1)$. In such a situation we will also say that C has the *trivial normal bundle T.*

THEOREM 9.1. *Every tube in a stable manifold is trivial.*

THEOREM 9.2. *Let M^n be a connected topological manifold and (T_i) a family of trivial tubes whose cores freely generate $\pi_1(M^n)$. Then M^n admits a stable structure.*

The main result is

THEOREM 9.3. *Every locally flat simple closed curve in a stable manifold M^n has a trivial normal bundle in M^n.*

It is easy to show that $\pi_1(M^n)$ can always be generated by locally flat simple closed curves. Then combining the above theorem with Theorem 9.2, we get the following characterizations of stable manifolds.

THEOREM 9.4. *M^n admits a stable structure if and only if $\pi_1(M^n)$ can be freely generated by trivial tubes.*

THEOREM 9.5. *M^n admits a stable structure if and only if every locally flat simple closed curve in M^n has a trivial normal bundle in M^n.*

10. Solution of the Schoenflies Problem for $S^{n-1}\times S^1$. If M_1 and M_2 are two n-manifolds, a *sum* $M_1 \# M_2$ is (ambiguously) obtained by removing the interiors of locally flat closed n-cells from M_1 and M_2 and attaching the new boundaries of the remaining manifolds to one another by some homeomorphism. $M_1 \# M_2$ is well-defined if all sums of M_1 and M_2 are homeomorphic. The manifold $M^n \# (S^{n-1}\times S^1)$ will be said to be obtained by adding a *handle* to M^n.

THEOREM 10.1. *Let M_1 and M_2 be connected topological n-manifolds. If either M_1 or M_2 is homogeneous, then $M_1 \# M_2$ is well-defined. If both are homogeneous, then $M_1 \# M_2$ is also homogeneous.*

The next theorem provides a solution of the Schoenflies Problem for $S^{n-1}\times S^1$.

THEOREM 10.2. *Let f_1 and f_2 be locally flat embeddings of S^{n-1} into $S^{n-1}\times S^1$ whose images either both separate or both do not separate $S^{n-1}\times S^1$. Then there is a homeomorphism h of $S^{n-1}\times S^1$ onto itself such that $hf_1=f_2$.*

The following is an easy corollary.

THEOREM 10.3. *$S^{n-1}\times S^1$ is homogeneous, and the stable structure on it is unique.*

Then by Theorem 10.1 we have

THEOREM 10.4. *The operation of adding a handle to a connected manifold is well-defined.*

THEOREM 10.5. *The n-sphere with k handles is well-defined and homogeneous, and the stable structure on it is unique.*

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